About Fall Colorings of Graphs

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Abstract

A fall k-coloring of a graph G is a proper k-coloring of G such that each vertex of G sees all k colors on its closed neighborhood. In this paper, we answer some questions of [5] about some relations between fall colorings and some other types of graph colorings.
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1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless and without multiple edges). Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$ and $[k] := \{i|\ i \in \mathbb{N},\ 1 \leq i \leq k\}$. A k-coloring (proper k-coloring) of G is a function $f : V \to [k]$ such that for each $1 \leq i \leq k$, $f^{-1}(i)$ is an independent set. We say that G is k-colorable whenever G admits a k-coloring f, in this case, we denote $f^{-1}(i)$ by V_i and call each $1 \leq i \leq k$, a color (of f) and each V_i, a color class (of f). The minimum integer k for which G has a k-coloring, is called the chromatic number of G and is denoted by $\chi(G)$.

Let G be a graph, f be a k-coloring of G and v be a vertex of G. The vertex v is called colorful (or color-dominating or b-dominating) if each color $1 \leq i \leq k$ appears on the closed neighborhood of v ($f(N[v]) = [k]$). The k-coloring f is said to be a fall k-coloring (of G) if each vertex of G is colorful. There are graphs G for which G has no fall k-coloring for any positive integer k. For example, C_5 (a cycle with 5 vertices) and graphs with at least one edge and one isolated vertex, have not any fall k-colorings for any positive integer k. The notation Fall(G) stands for the set of all positive integers k for which G has a fall k-coloring. Whenever Fall(G) $\neq \emptyset$, we call min(Fall(G)) and max(Fall(G)), fall chromatic number of G and fall achromatic number of G and denote them by $\chi_f(G)$ and $\psi_f(G)$, respectively. Every fall k-coloring of a graph G is a k-coloring, hence, for every graph G with Fall(G) $\neq \emptyset$, $\chi(G) \leq \chi_f(G) \leq \psi_f(G)$.

Let G be a graph, $k \in \mathbb{N}$ and f be a k-coloring of G. The coloring f is said to be a colorful k-coloring of G if each color class contains at least one colorful vertex. The maximum integer k for which G has a colorful k-coloring, is called the b-chromatic number of G and is denoted by $\phi(G)$ (or $b(G)$ or $\chi_b(G)$). Every fall k-coloring of G is obviously a colorful k-coloring of G and therefore, for every graph G with Fall(G) $\neq \emptyset$, $\chi(G) \leq \chi_f(G) \leq \psi_f(G) \leq \phi(G)$.

Assume that G is a graph, $k \in \mathbb{N}$ and f is a k-coloring of G and v be a vertex of G. The vertex v is called a Grundy vertex (with respect to f) if each color $1 \leq i < f(v)$

appears on the neighborhood of v. The k-coloring f is called a Grundy k-coloring (of G) if each color class of G is nonempty and each vertex of G is a Grundy vertex. The maximum integer k for which G has a Grundy k-coloring, is called the Grundy chromatic number of G and is denoted by $\Gamma(G)$. Also, the k-coloring f is said to be a partial Grundy k-coloring (of G) if each color class contains at least one Grundy vertex. The maximum integer k for which G has a partial Grundy k-coloring, is called the partial Grundy chromatic number of G and is denoted by $\partial\Gamma(G)$. Every Grundy k-coloring of G is a partial Grundy k-coloring of G and every colorful k-coloring of G is a partial Grundy k-coloring of G. Also, every fall k-coloring of G is obviously a Grundy k-coloring of G and therefore, for every graph G with $\mathrm{Fall}(G) \neq \emptyset$,

$$\chi(G) \leq \chi_f(G) \leq \psi_f(G) \leq \left\{ \begin{array}{c} \phi(G) \\ \Gamma(G) \end{array} \right. \leq \partial\Gamma(G).$$

Let G be a graph and $k \in \mathbb{N}$ and f be a k-coloring of G. The k-coloring f is said to be a complete k-coloring (of G) if there is an edge between any two distinct color classes. The maximum integer k for which G has a complete k-coloring, is called the achromatic number of G and is denoted by $\psi(G)$. Every partial Grundy k-coloring of G is obviously a complete k-coloring of G and therefore,

$$\chi(G) \leq \chi_f(G) \leq \psi_f(G) \leq \left\{ \begin{array}{c} \phi(G) \\ \Gamma(G) \end{array} \right. \leq \partial\Gamma(G) \leq \psi(G).$$

The terminology fall coloring was firstly introduced in 2000 in [5] and has received attention recently, see [3],[4],[5],[8]. The colorful coloring of graphs was introduced in 1999 in [7] with the terminology b-coloring. The concept of Grundy number of graphs was introduced in 1979 in [1]. Also, achromaric number of graphs was introduced in 1970 in [6].

Let $n \in \mathbb{N}$ and for each $1 \leq i \leq n$, G_i be a graph. The graph with vertex set $\bigcup_{i=1}^{n}(\{i\} \times V(G_i))$ and edge set

$$[\bigcup_{i=1}^{n}\{\{(i,x),(i,y)\}|\{x,y\} \in E(G_i)\}]\bigcup[\bigcup_{1\leq i<j\leq n}\{\{(i,a),(j,b)\}|a \in V(G_i), b \in V(G_j)\}]$$

is called the join graph of $G_1, ..., G_n$ and is denoted by $\bigvee_{i=1}^{n} G_i$.

Cockayne and Hedetniemi proved in 1976 in [2] (but not with the terminology "fall coloring") that if G has a fall k-coloring and H has a fall l-coloring for positive integers k and l, then, $G \bigvee H$ has a fall $(k+l)$-coloring.

Theorem 1. *Let $n \in \mathbb{N} \setminus \{1\}$ and for each $1 \leq i \leq n$, G_i be a graph. Then,*
$\mathrm{Fall}(\bigvee_{i=1}^{n} G_i) \neq \emptyset$ *iff for each $1 \leq i \leq n$,* $\mathrm{Fall}(G_i) \neq \emptyset$.

Proof. First suppose that $\mathrm{Fall}(\bigvee_{i=1}^{n} G_i) \neq \emptyset$. Consider a fall k-coloring f of $\bigvee_{i=1}^{n} G_i$. The colors appear on $\{i\} \times V(G_i)$ form a fall $|f(\{i\} \times V(G_i))|$-coloring of G_i (Let S be the set of colors appear on $\{i\} \times V(G_i)$ and $\alpha, \beta \in S$ and $\alpha \neq \beta$ and $x \in \{i\} \times V(G_i)$ and $f(x) = \alpha$ and suppose that none of the neighbors of x in $\{i\} \times V(G_i)$ have the color β. Since f is a fall k-coloring of $\bigvee_{i=1}^{n} G_i$, there exists a vertex $y \in V(\bigvee_{i=1}^{n} G_i)$ such that $\{x,y\} \in E(\bigvee_{i=1}^{n} G_i)$ and $f(y) = \beta$. On the other hand, since $\beta \in S$, there exists a vertex z in $\{i\} \times V(G_i)$ such that $f(z) = \beta$. Since x and y are adjacent, z and y are adjacent, too. Also, $f(z) = f(y) = \beta$, which

is a contradiction. Therefore, S forms a $|S|$-coloring of the induced subgraph of $\bigvee_{i=1}^{n} G_i$ on $\{i\} \times V(G_i)$ and also for G_i.) and therefore, $\mathrm{Fall}(G_i) \neq \emptyset$. Conversely, suppose that for each $1 \leq i \leq n$, $k_i \in \mathrm{Fall}(G_i)$. For each $1 \leq i \leq n$, construct a fall k_i-coloring of the induced subgraph of $\bigvee_{i=1}^{n} G_i$ on $\{i\} \times V(G_i)$ with the color set $\{(\sum_{j=1}^{i-1} k_j)+1, (\sum_{j=1}^{i-1} k_j)+2, ..., (\sum_{j=1}^{i-1} k_j)+k_i\}$. This forms a fall $(\sum_{i=1}^{n} k_i)$-coloring of $\bigvee_{i=1}^{n} G_i$ and therefore, $\mathrm{Fall}(\bigvee_{i=1}^{n} G_i) \neq \emptyset$. ■

The proof of the following obvious theorem has omitted for the sake of brevity.

Theorem 2. *Let $n \in \mathbb{N} \setminus \{1\}$ and for each $1 \leq i \leq n$, G_i be a graph. Then,*

1)If for each $1 \leq i \leq n$, $\mathrm{Fall}(G_i) \neq \emptyset$, then, $\mathrm{Fall}(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \mathrm{Fall}(G_i) := \{a_1 + ... + a_n |\ a_1 \in \mathrm{Fall}(G_1), ...,\ a_n \in \mathrm{Fall}(G_n)\}$ and $\chi_f(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \chi_f(G_i)$ and $\psi_f(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \psi_f(G_i)$.

2) $\chi(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \chi(G_i)$.

3) $\phi(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \phi(G_i)$.

4) $\Gamma(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \Gamma(G_i)$.

5) $\partial\Gamma(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \partial\Gamma(G_i)$.

6) $\psi(\bigvee_{i=1}^{n} G_i) = \sum_{i=1}^{n} \psi(G_i)$.

In [5], Dunbar, et al. asked the following questions.

1*) whether or not there exists a graph G with $\mathrm{Fall}(G) \neq \emptyset$ which satisfies $\chi_f(G) - \chi(G) \geq 3$? They noticed that $\chi_f(C_4 \Box C_5) = 4$ and $\chi(C_4 \Box C_5) = 3$, also, $\chi_f(C_5 \Box C_5) = 5$ and $\chi(C_5 \Box C_5) = 3$.

2*) Can $\chi_f(G) - \chi(G)$ be arbitrarily large?

3*) Does there exist a graph G with Fall(G) $\neq \emptyset$ which satisfies $\chi(G) < \chi_f(G) < \psi_f(G) < \phi(G) < \partial\Gamma(G) < \psi(G)$?

Since $\chi_f(C_4 \Box C_5) = 4$ and $\chi(C_4 \Box C_5) = 3$, Theorem 2 implies that For each $n \in \mathbb{N}$, $\chi_f(\bigvee_{i=1}^{n}(C_4 \Box C_5)) - \chi(\bigvee_{i=1}^{n}(C_4 \Box C_5)) = 4n - 3n = n$ and this gives an affirmative answer to the problems 1* and 2*.

Also, the Theorem 2 and the following theorem, give an affirmative answer to all 3 questions immediately.

Theorem 3. *For each integer $\varepsilon > 0$, there exists a graph G with $Fall(G) \neq \emptyset$ which the minimum of $\chi_f(G) - \chi(G)$, $\psi_f(G) - \chi_f(G)$, $(\delta(G)+1) - \psi_f(G)$, $\Gamma(G) - \psi_f(G)$, $\phi(G) - \psi_f(G)$, $(\Delta(G)+1) - \partial\Gamma(G)$, $\psi(G) - \partial\Gamma(G)$, $\partial\Gamma(G) - \Gamma(G)$ is greater than ε.*

Proof. Let $\varepsilon > 2$ be an arbitrary integer and let's follow the following steps.

Step1) Set $G_1 := \bigvee_{i=1}^{\varepsilon+1}(C_4 \Box C_5)$. As stated above, $\chi_f(G_1) - \chi(G_1) = \varepsilon + 1$.

Step2) Set $G_2 := K_{\varepsilon+3,\varepsilon+3} - ($ an arbitrary $1 - factor$ $)$. One can easily observe that $\psi_f(G_2) - \chi_f(G_2) = (\varepsilon + 3) - 2 = \varepsilon + 1$.

Step3) Set $G_3 := K_{(\varepsilon+2,\varepsilon+2)}$. Then, $(\delta(G_3)+1)-\psi_f(G_3) = ((\varepsilon+2)+1)-2 = \varepsilon+1$.

Step4) Let $P_{\varepsilon+3}$ be a path with $\varepsilon+3$ vertices. Add $\varepsilon+2$ pendant vertices to each of its vertices and denote the new graph by G_4. It is readily seen that $\phi(G_4) - \psi_f(G_4) \geq (\varepsilon+3) - 2 = \varepsilon+1$.

Step5) Let $T(1)$ be the tree with only one vertex and for each $k \geq 1$, $T(k+1)$ be the graph obtained by adding a new pendant vertex to each vertex of $T(k)$. $T(\varepsilon+3)$ is a tree which its Grundy number is $\varepsilon+3$ and $\psi_f(T(\varepsilon+3)) \leq \delta(T(\varepsilon+3)) + 1 \leq 2$. Hence, if we set $G_5 := T(\varepsilon+3)$, then, $\Gamma(G_5) - \psi_f(G_5) = \varepsilon+1$.

Step6) Let G_6 be the graph obtained by adding $i-2$ pendant vertices to each vertex $v_i(3 \leq i \leq \varepsilon+5)$ of the path $v_1v_2\ldots v_{\varepsilon+5}$. Obviously, $\partial\Gamma(G_6) \geq \varepsilon+5$ and $\Gamma(G_6) \leq 4$. So, $\partial\Gamma(G_6) - \Gamma(G_6) \geq \varepsilon+1$.

Step7) Set $G_7 := K_{\varepsilon+2,\varepsilon+2}$. $(\Delta(G_7)+1) - \partial\Gamma(G_7) = (\varepsilon+3) - 2 = \varepsilon+1$.

Step8) Set $G_8 := P_{\frac{(\varepsilon+4)(\varepsilon+3)}{2}}$. Obviously, $\psi(G_8) \geq \varepsilon+4$ and $\partial\Gamma(G_8) \leq \triangle(G_8) + 1 \leq 3$. Hence, $\psi(G_8) - \partial\Gamma(G_8) \geq \varepsilon+1$.

Step9) Set $G := \bigvee_{i=1}^{8} G_i$. For each $1 \leq i \leq 8$, $\mathrm{Fall}(G_i) \neq \emptyset$. Hence, by Theorem 2, the fact that $\delta(\bigvee_{i=1}^{8} G_i) \geq \sum_{i=1}^{8} \delta(G_i)$, and $\Delta(\bigvee_{i=1}^{8} G_i) \geq \sum_{i=1}^{8} \Delta(G_i)$, G_9 is a suitable graph for this theorem and also for questions 1*, 2* and 3*.

■

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